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On the curvature of Kähler–Norden manifolds

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Abstract

Using the one-to-one correspondence between Kähler–Norden and holomorphic Riemannian metrics, important relations between various Riemannian invariants of manifolds endowed with such metrics are established. Especially, the holomorphic versions of the recurrence of the Riemann, Ricci, projective are defined and investigated. For four-dimensional Kähler–Norden manifolds, it is proved that they are of holomorphically recurrent curvature on the set where the holomorphic scalar curvature does not vanish. Furthermore, a four-dimensional Kähler–Norden manifold is (locally) conformally flat if and only if its holomorphic scalar curvature is constant pure imaginary. The present paper continues author's investigations of Kähler–Norden manifolds from the papers [K. Słuka, On Kähler manifolds with Norden metrics, An. Ştiint. Univ. Al.I. Cuza IaşI Ser. Ia Mat. 47 (2001) 105–122; K. Słuka, Properties of the Weyl conformal curvature of Kähler–Norden manifolds, in: Proc. Colloq. Diff. Geom. on Steps in Differential Geometry, July 25–30, 2000, Debrecen, 2001, pp. 317–328].

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1. Preliminaries

1.1. Kähler–Norden manifolds

Let M be a real connected $2m$-dimensional differentiable manifold endowed with an almost complex structure J ($J^2 = -I$, I being the identity transformation) and a pseudo-Riemannian metric g of Norden type (that is, of signature (m, m)) and such that

$$g(JX, JY) = -g(X, Y), \tag{1}$$

$$(\nabla_X J)Y = 0 \tag{2}$$

for any $X, Y \in \mathfrak{X}(M)$, where ∇ is the Levi–Civita connection of g and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. Then the triple (M, J, g) will be called a Kähler–Norden manifold (a Kählerian manifold with Norden metric [5], an anti-Kählerian manifold [1,2]).

1.2. Holomorphic Riemannian manifolds

Let M be a complex manifold of complex dimension m. Denote by (M, J) the manifold considered as a real $2m$-dimensional manifold with the induced almost complex structure J. The tangent space to (M, J) at $p \in M$ and its complexification are denoted by T_pM and $T_p^{\mathbb{C}}M$, respectively. The subspaces of $T_p^{\mathbb{C}}M$ consisting of complex vectors of type (1, 0) and (0, 1) are denoted by $T_p^{(1,0)}M$ and $T_p^{(0,1)}M$, respectively. The Lie algebras of real smooth vector fields, complex vector fields, complex vector fields of type (1, 0) and complex vector fields of type (0, 1) on M are denoted by $\mathfrak{X}(M)$, $\mathfrak{X}^{\mathbb{C}}(M)$, $\mathfrak{X}^{(1,0)}(M)$ and $\mathfrak{X}^{(0,1)}(M)$, respectively.

By a complex Riemannian metric on M is meant a symmetric (0, 2)-tensor field G, which is non-degenerate at each point of M and

$$G(\bar{Z}_1, \bar{Z}_2) = \overline{G(Z_1, Z_2)} \quad \text{for any} \quad Z_1, Z_2 \in \mathfrak{X}^{\mathbb{C}}(M),$$
$$G(Z_1, Z_2) = 0 \quad \text{for any} \quad Z_1 \in \mathfrak{X}^{(1,0)}(M) \quad \text{and} \quad Z_2 \in \mathfrak{X}^{(0,1)}(M). \tag{3}$$

The second condition of (3) is equivalent to

$$G(JZ_1, JZ_2) = -G(Z_1, Z_2) \quad \text{for any} \quad Z_1, Z_2 \in \mathfrak{X}^{\mathbb{C}}(M).$$

Thus, a complex Riemannian metric is completely determined by its values on $\mathfrak{X}^{(1,0)}(M)$.

If G is a complex Riemannian metric on a complex manifold M, then the pair (M, G) is said to be a complex Riemannian manifold [6,7,9,22].

For a local holomorphic coordinates system $(z^\alpha; 1 \le \alpha \le m)$ of a complex Riemannian manifold, let $z^\alpha = x^\alpha + \sqrt{-1}y^\alpha$, where $x^\alpha = \mathrm{Re}(z^\alpha)$, $y^\alpha = \mathrm{Im}(z^\alpha)$, and suppose

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - \sqrt{-1}\frac{\partial}{\partial y^\alpha}\right), \qquad \frac{\partial}{\partial \bar{z}^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + \sqrt{-1}\frac{\partial}{\partial y^\alpha}\right).$$

In terms of such local coordinates, we set

$$G_{AB} = G\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right), \quad A, B = 1, \ldots, m, \bar{1}, \ldots, \bar{m}.$$

Then, for a complex Riemannian metric G, the defining conditions (3) are equivalent to

$$G_{\bar{\alpha}\bar{\beta}} = \overline{G_{\alpha\beta}}, \quad G_{\bar{\alpha}\beta} = G_{\alpha\bar{\beta}} = 0.$$

A complex Riemannian manifold (M, G) is said to be a holomorphic Riemannian manifold (see [6,7]; and also [4,12,13,22]) if the local components $G_{\alpha\beta}$ are holomorphic functions, that is:

$$\frac{\partial}{\partial z^{\bar{\gamma}}} G_{\alpha\beta} = 0. \tag{4}$$

The condition (4) is equivalent to $\hat{\nabla} J = 0$, where $\hat{\nabla}$ is the Levi–Civita connection of G [6].

2. Kähler–Norden versus holomorphic Riemannian

In [1,2], it was shown that there is one-to-one correspondence between Kähler–Norden manifolds (M, J, g) and holomorphic Riemannian manifolds (M, G). The reader could also confer [22]. We will describe this correspondence as follows, referring to [1,2] for details.

Let (M, J, g) be a Kähler–Norden manifold, so that the conditions (1) and (2) are fulfilled. Since the almost complex structure J is parallel with respect to the connection ∇, it is integrable. Therefore, the real manifold M inherits the structure of a complex manifold, which for simplicity's sake will also be denoted by M, and the almost complex structure J comes from the complex structure in the usual way.

In this paper, for any $X \in \mathfrak{X}(M)$, by $\hat{X}$ we always denote the complex vector field of type (1, 0) generated by X, that is:

$$\hat{X} = \tfrac{1}{2}(X - \sqrt{-1}\, JX) \in \mathfrak{X}^{(1,0)}(M).$$

Any vector field $Z \in \mathfrak{X}^{(1,0)}(M)$ is of this form, i.e., $Z = \hat{X}$ for a certain $X \in \mathfrak{X}(M)$ (see e.g. [11], vol. II).

To define a complex Riemannian metric G on the complex manifold M it is sufficient to suppose

$$G(\hat{X}, \hat{Y}) = \tfrac{1}{2}(g(X, Y) - \sqrt{-1}\, g(X, JY)), \quad X, Y \in \mathfrak{X}(M) \tag{5}$$

and next extend G to have the conditions (3) satisfied (it is possible because of (1)). Moreover, by (2) the metric G is holomorphic. Thus, (M, G) is a holomorphic Riemannian manifold.

Conversely, a holomorphic Riemannian manifold (M, G) can be considered as a real $2m$-dimensional Kähler–Norden manifold (M, J, g). Namely, we define J to be the almost complex structure coming from the complex structure of M and suppose

$$g(X, Y) = 2\,\mathrm{Re}(G(\hat{X}, \hat{Y})), \quad X, Y \in \mathfrak{X}(M). \tag{6}$$

Then (3) and (4) imply (1) and (2), respectively.

One easily checks that the relations (5) and (6) state the one-to-one correspondence between Kähler–Norden structures (J, g) and holomorphic Riemannian metrics G on M.

3. General formulas

Let (M, J, g) be a Kähler–Norden manifold and let $(M, \hat{g})$ be the corresponding holomorphic Riemannian manifold (in the sense explained in Section 2). Here and in the rest of this paper, we write $\hat{g}$ instead of G.

Let $\mathfrak{X}^h(M)$ denote the Lie algebra of holomorphic vector fields on M.

3.1. Agreement

Throughout the rest of this paper, without loss of generality, $X, Y, \ldots$ will denote arbitrary real smooth vector fields on M such that $\hat{X}, \hat{Y}, \ldots \in \mathfrak{X}^h(M)$.

The considered vector fields on M are always infinitesimal automorphisms of the almost complex structure J (cf. e.g. [11], vol. II). Therefore, we have

$$[JX, Y] = [X, JY] = J[X, Y], \qquad [JX, JY] = -[X, Y], \tag{7}$$

$$[\hat{X}, \hat{Y}] = \widehat{[X, Y]}. \tag{8}$$

One notes that for a holomorphic function f and a vector field $\hat{W}$, the following formula is valid

$$f\hat{W} = ((\mathrm{Re}\, f)W + (\mathrm{Im}\, f)JW)\hat{}\,. \tag{9}$$

Moreover, if f is a holomorphic function, then by the Cauchy–Riemann equations, we have the following useful formulas

$$X(\mathrm{Re}\, f) = (JX)(\mathrm{Im}\, f), \qquad (JX)(\mathrm{Re}\, f) = -X(\mathrm{Im}\, f), \tag{10}$$

$$\hat{X} f = X(\mathrm{Re}\, f) + \sqrt{-1}\, X(\mathrm{Im}\, f). \tag{11}$$

By $(e_1, e_2, \ldots, e_{2m})$ we denote a frame of a tangent space T_pM, which is adapted to the structure (J, g) in the sense that it consists of real vectors, such that $g(e_\alpha, e_\beta) = -g(e_{\alpha'}, e_{\beta'}) = \delta_{\alpha\beta}$, $g(e_\alpha, e_{\beta'}) = g(e_{\alpha'}, e_\beta) = 0$, $Je_\alpha = e_{\alpha'}$, $Je_{\alpha'} = -e_\alpha$, where the Greek indices take on values $1, \ldots, m$ and $\alpha' = \alpha + m$. Then assuming $\hat{e}_\alpha = (1/2)(e_\alpha - \sqrt{-1}\, Je_\alpha)$, we have also a frame $(\hat{e}_1, \ldots, \hat{e}_m)$ of the space $T_p^{(1,0)}M$ for which $\hat{g}(\hat{e}_\alpha, \hat{e}_\beta) = (1/2)\delta_{\alpha\beta}$.

Let ∇ and $\hat{\nabla}$ be the Levi–Civita connections of the Kähler–Norden metric g and the holomorphic Riemannian metric $\hat{g}$, respectively. The connection $\hat{\nabla}$ is holomorphic, that is, $\hat{\nabla}_{\hat{X}}\hat{Y} \in \mathfrak{X}^h(M)$ for any $\hat{X}, \hat{Y} \in \mathfrak{X}^h(M)$ (cf. [12,13,4]). By the symmetry of ∇, one notes the following important consequence of (7):

$$\nabla_{JX} Y = J\nabla_X Y. \tag{12}$$

To establish the basic relation between the Levi–Civita connections ∇ and $\hat{\nabla}$, we apply the standard formulas (cf. e.g. [11], vol. I)

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g([X, Z], Y) - g([Y, Z], X) + g([X, Y], Z), \tag{13}$$

$$2\hat{g}(\hat{\nabla}_{\hat{X}}\hat{Y}, \hat{Z}) = \hat{X}\hat{g}(\hat{Y}, \hat{Z}) + \hat{Y}\hat{g}(\hat{X}, \hat{Z}) - \hat{Z}\hat{g}(\hat{X}, \hat{Y}) - \hat{g}([\hat{X}, \hat{Z}], \hat{Y}) - \hat{g}([\hat{Y}, \hat{Z}], \hat{X}) + \hat{g}([\hat{X}, \hat{Y}], \hat{Z}). \tag{14}$$

Using (11), (5), (8) and (10) we find

$$\hat{X}\hat{g}(\hat{Y}, \hat{Z}) = \tfrac{1}{2}(Xg(Y, Z) - \sqrt{-1}\, Xg(Y, JZ)),$$

$$\hat{g}([\hat{X}, \hat{Y}], \hat{Z}) = \tfrac{1}{2}(g([X, Y], Z) - \sqrt{-1}\, g([X, Y], JZ)), \quad Zg(X, JY) = (JZ)g(X, Y).$$

Using the above formulas, (7) and (13), the right hand side of (14) can be transformed into

$$g(\nabla_X Y, Z) - \sqrt{-1} g(\nabla_X Y, JZ)$$

and next by (5) into $2\hat{g}(\widehat{\nabla_X Y}, \hat{Z})$. This compared to the left hand side of (14), gives the desired formula

$$\hat{\nabla}_{\hat{X}}\hat{Y} = \widehat{\nabla_X Y}. \tag{15}$$

Let R and $\hat{R}$ be the Riemann curvature tensors of ∇ and $\hat{\nabla}$, respectively:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \qquad \hat{R}(\hat{X}, \hat{Y}) = [\hat{\nabla}_{\hat{X}}, \hat{\nabla}_{\hat{Y}}] - \hat{\nabla}_{[\hat{X},\hat{Y}]}.$$

By applying (7), (12) and $\nabla J = 0$, we can show that R is totally pure, i.e.:

$$R(X, Y)J = R(JX, Y) = R(X, JY) = JR(X, Y). \tag{16}$$

Using the defining formulas and (15) and (8), we check by straightforward computations that the Riemann curvature tensors R and $\hat{R}$ are related by

$$\hat{R}(\hat{X}, \hat{Y})\hat{Z} = (R(X, Y)Z)\hat{}. \tag{17}$$

Let S and $\hat{S}$ be the Ricci curvature tensor fields:

$$S(X, Y) = \mathrm{Tr}\{Z \mapsto R(Z, X)Y\}, \qquad \hat{S}(\hat{X}, \hat{Y}) = \mathrm{Tr}\{\hat{Z} \mapsto \hat{R}(\hat{Z}, \hat{X})\hat{Y}\}$$

and let Q and $\hat{Q}$ be the corresponding Ricci operators:

$$g(QX, Y) = S(X, Y), \qquad \hat{g}(\hat{Q}\hat{X}, \hat{Y}) = \hat{S}(\hat{X}, \hat{Y}).$$

For the Ricci curvature tensor S and the Ricci operator Q, we have

$$S(JX, Y) = S(X, JY), \qquad S(JX, JY) = -S(X, Y), \qquad QJ = JQ. \tag{18}$$

In fact, using (16), we find

$$S(JX, Y) = \mathrm{Tr}\{Z \mapsto R(Z, JX)Y\} = \mathrm{Tr}\{Z \mapsto R(Z, X)JY\} = S(X, JY).$$

Hence, it easily follows that $S(JX, JY) = -S(X, Y)$. Moreover, the first equality of (18) leads to $g(QJX, Y) = g(QX, JY) = g(JQX, Y)$, which gives $QJ = JQ$.

Note that for S, $\hat{S}$, Q and $\hat{Q}$, it holds

$$\hat{S}(\hat{X}, \hat{Y}) = \tfrac{1}{2}(S(X, Y) - \sqrt{-1}\, S(X, JY)), \qquad \hat{Q}\hat{X} = \widehat{QX}. \tag{19}$$

Indeed, using the adapted frame (e_i) and formula (16), we can write

$$S(X, Y) = \sum (g(R(e_\alpha, X)Y, e_\alpha) - g(R(e_{\alpha'}, X)Y, e_{\alpha'})) = 2 \sum (g(R(e_\alpha, X)Y, e_\alpha),$$

where the sum concerns the repeated indices. Next, using the frame $(\hat{e}_\alpha)$ and formulas (17), (5) and (16), we find

$$\begin{aligned}\hat{S}(\hat{X}, \hat{Y}) &= 2 \sum \hat{g}(\hat{R}(\hat{e}_\alpha, \hat{X})\hat{Y}, \hat{e}_\alpha) = 2 \sum \hat{g}((R(e_\alpha, X)Y)\hat{}, \hat{e}_\alpha) \\ &= \sum (g(R(e_\alpha, X)Y, e_\alpha) - \sqrt{-1}\, g(R(e_\alpha, X)Y, Je_\alpha)) \\ &= \sum (g(R(e_\alpha, X)Y, e_\alpha) - \sqrt{-1}\, g(R(e_\alpha, X)JY, e_\alpha)).\end{aligned}$$

The above together with the previous equality implies the first part of (19). This together with (5) gives for the Ricci operators Q and $\hat{Q}$

$$\begin{aligned}\hat{g}(\hat{Q}\hat{X}, \hat{Y}) &= \hat{S}(\hat{X}, \hat{Y}) = \tfrac{1}{2}(S(X, Y) - \sqrt{-1}\, S(X, JY)) \\ &= \tfrac{1}{2}(g(QX, Y) - \sqrt{-1}\, g(QX, JY)) = \hat{g}(\widehat{QX}, \hat{Y}),\end{aligned}$$

completing the proof of (19).

Define the real scalar curvatures r, r^* of g, and the holomorphic scalar curvature $\hat{r}$ of $\hat{g}$ by

$$r = \mathrm{Tr}\, Q, \qquad r^* = \mathrm{Tr}(JQ), \qquad \hat{r} = \mathrm{Tr}\, \hat{Q}.$$

Using (18) and (1), we find the following expressions for r and r^*:

$$\begin{aligned}r &= \mathrm{Tr}\, Q = \sum (g(Qe_\alpha, e_\alpha) - g(Qe_{\alpha'}, e_{\alpha'})) = 2 \sum (g(Qe_\alpha, e_\alpha)), \\ r^* &= \mathrm{Tr}(JQ) = \sum (g(JQe_\alpha, e_\alpha) - g(JQe_{\alpha'}, e_{\alpha'})) = 2 \sum (g(Qe_\alpha, Je_\alpha)).\end{aligned}$$

Moreover, using (19) and (5), we obtain for $\hat{r}$:

$$\begin{aligned}\hat{r} &= \mathrm{Tr}\, \hat{Q} = 2 \sum \hat{g}(\hat{Q}\hat{e}_\alpha, \hat{e}_\alpha) = 2 \sum \hat{g}(\widehat{Qe_\alpha}, \hat{e}_\alpha) \\ &= \sum (g(Qe_\alpha, e_\alpha) - \sqrt{-1}\, g(Qe_\alpha, Je_\alpha)).\end{aligned}$$

From the above equalities, one can easily derive the following important formula for $\hat{r}$:

$$\hat{r} = \tfrac{1}{2}(r - \sqrt{-1}\, r^*). \tag{20}$$

For further use, we need also the following additional operators.

For $X, Y \in \mathfrak{X}(M)$ and a symmetric (0, 2)-tensor field A on M, define $X \wedge_A Y$ to be the operator acting on vector fields by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \quad Z \in \mathfrak{X}(M).$$

Let A satisfy additionally the condition

$$A(JX, JY) = -A(X, Y)$$

(note that the relation of this type is fulfilled by the metric tensor g and the Ricci tensor S; cf. (1) and (18)).

Define $\hat{A}$ to be the complex (0, 2)-tensor field which is completely determined by its values on $\mathfrak{X}^{(1,0)}(M)$ and for which

$$\hat{A}(\hat{X}, \hat{Y}) = \tfrac{1}{2}(A(X, Y) - \sqrt{-1}\, A(X, JY)).$$

For $\hat{X}, \hat{Y} \in \mathfrak{X}^{(1,0)}(M)$, define the operator $\hat{X} \wedge_{\hat{A}} \hat{Y}$ acting on $\mathfrak{X}^{(1,0)}(M)$ by

$$(\hat{X} \wedge_{\hat{A}} \hat{Y})\hat{Z} = \hat{A}(\hat{Y}, \hat{Z})\hat{X} - \hat{A}(\hat{X}, \hat{Z})\hat{Y}.$$

It is now a straightforward verification that

$$(\hat{X} \wedge_{\hat{A}} \hat{Y})\hat{Z} = \tfrac{1}{2}((X \wedge_A Y)Z - (JX \wedge_A JY)Z)\hat{}. \tag{21}$$

If additionally $\hat{A}$ is a holomorphic tensor field, then $(\hat{X} \wedge_{\hat{A}} \hat{Y})\hat{Z} \in \mathfrak{X}^h(M)$ for any $\hat{X}, \hat{Y}, \hat{Z} \in \mathfrak{X}^h(M)$.

4. Recurrence of the Riemann curvature

A (pseudo-)Riemannian manifold is said to be (1) locally symmetric if $\nabla R = 0$; (2) of recurrent curvature [11,21] if its Riemann curvature tensor R is not identically zero and for a certain 1-form ψ, $\nabla R = \psi \otimes R$. It was shown by the author in [18] that the recurrence of the Riemann curvature of a Kähler–Norden manifold is not essential in the sense such a manifold is necessarily locally symmetric. On the other hand, the following notion of the holomorphic recurrence is crucial.

A Kähler–Norden manifold will be said to be of holomorphically recurrent (in short, H-recurrent) curvature if its holomorphic Riemann curvature tensor $\hat{R}$ is not identically zero and for a certain holomorphic 1-form $\hat{\varphi}$:

$$\hat{\nabla}\hat{R} = \hat{\varphi} \otimes \hat{R}. \tag{22}$$

Before we express the holomorphic recurrence with the help of the real Riemann curvature tensor R, we extend J to act on real 1-forms φ assuming $(J\varphi)(X) = \varphi(JX)$ for any $X \in \mathfrak{X}(M)$.

Remark 1. J is compatible with the musical isomorphisms. Precisely, let $\sharp : T^*M \to TM$ and $\flat : TM \to T^*M$ be the isomorphisms between tangent and cotangent bundles TM, T^*M defined by $g(\omega^\sharp, X) = \omega(X)$ and $X^\flat(Y) = g(X, Y)$ for any $\omega \in T^*M$ and $X, Y \in TM$. With the help of (1), one easily checks that J is compatible with $\sharp$ and $\flat$, that is, $(J\omega)^\sharp = J\omega^\sharp$ and $(JX)^\flat = JX^\flat$.

Theorem 2. *A Kähler–Norden manifold is of H-recurrent curvature if and only if the curvature tensor R is not identically zero and*

$$(\nabla_U R)(X, Y) = \varphi(U)R(X, Y) - \varphi(JU)JR(X, Y) \tag{23}$$

for a certain real 1-form φ such that $\hat{\varphi} = \varphi - \sqrt{-1}J\varphi$ is a holomorphic 1-form.

Proof. At first, we observe that for the covariant derivatives $\hat{\nabla}\hat{R}$ and ∇R of a Kähler–Norden manifold:

$$(\hat{\nabla}_{\hat{U}}\hat{R})(\hat{X}, \hat{Y})\hat{Z} = ((\nabla_U R)(X, Y)Z)\hat{}. \tag{24}$$

To prove it, we write down the standard formula

$$(\hat{\nabla}_{\hat{U}}\hat{R})(\hat{X}, \hat{Y})\hat{Z} = \hat{\nabla}_{\hat{U}}(\hat{R}(\hat{X}, \hat{Y})\hat{Z}) - \hat{R}(\hat{\nabla}_{\hat{U}}\hat{X}, \hat{Y})\hat{Z} - \hat{R}(\hat{X}, \hat{\nabla}_{\hat{U}}\hat{Y})\hat{Z} - \hat{R}(\hat{X}, \hat{Y})\hat{\nabla}_{\hat{U}}\hat{Z}.$$

An application of the formulas (15) and (17) enables us to rewrite the right hand side of the above in the form $((\nabla_U R)(X, Y)Z)\hat{}$. Now we can start the current proof. Writing a holomorphic 1-form $\hat{\varphi}$ as $\hat{\varphi} = \varphi - \sqrt{-1}J\varphi$, with φ being a certain real 1-form, we see that

$$\hat{\varphi}(\hat{U}) = \varphi(U) - \sqrt{-1}\,\varphi(JU). \tag{25}$$

The formulas (17), (25) and (9) with $f = \hat{\varphi}(\hat{U})$, $W = R(X, Y)Z$ enable us to find the following

$$\hat{\varphi}(\hat{U})\hat{R}(\hat{X}, \hat{Y})\hat{Z} = \hat{\varphi}(\hat{U})(R(X, Y)Z)\hat{} = (\varphi(U)R(X, Y)Z - \varphi(JU)JR(X, Y)Z)\hat{}.$$

Now using the above expression and (24), we claim that the defining condition (22) is equivalent to (23). □

Examples of Kähler–Norden manifolds of H-recurrent curvature will be given in the last section. They are not locally symmetric in general.

5. Ricci-recurrence

A (pseudo-)Riemannian manifold is said to be of recurrent Ricci curvature [16] if its Ricci curvature tensor S is not identically zero and for a certain 1-form ψ:

$$\nabla S = \psi \otimes S. \tag{26}$$

Proposition 3. *Any Kähler–Norden manifold of recurrent Ricci curvature is necessarily of parallel Ricci curvature.*

Proof. In the proof, we need the following formula for ∇S, which is valid for any Kähler–Norden manifold:

$$(\nabla_{JU} S)(X, Y) = (\nabla_U S)(X, JY). \tag{27}$$

In fact, using (12), we find $(\nabla_{JU} Q)X = J(\nabla_U Q)X$, from which the expression (27) follows.

Let the manifold be of recurrent Ricci curvature. Recall that under our assumption, the Ricci tensor S does not vanish at any point of M. Suppose that the set of points at which the recurrence form $\psi \neq 0$ is not empty and restrict our consideration to this set. By applying (26) into (27), we have $\psi(JU)S(X, Y) - \psi(U)S(X, JY) = 0$. Hence, $S(X, Y)J\psi - S(X, JY)\psi = 0$. Since ψ and $J\psi$ are linearly independent, we find $S(X, Y) = 0$ for any X, Y, which is a contradiction. Therefore $\psi = 0$ at every point of M, that is, the Ricci tensor S is parallel. □

Because of the reduction appeared in Proposition 3, we propose to study the holomorphic Ricci-recurrence.

A Kähler–Norden manifold will be called of H-recurrent Ricci curvature if its holomorphic Ricci curvature tensor $\hat{S}$ is not identically zero and for a certain holomorphic 1-form $\hat{\varphi}$:

$$\hat{\nabla}\hat{S} = \hat{\varphi} \otimes \hat{S}. \tag{28}$$

Theorem 4. *A Kähler–Norden manifold is of H-recurrent Ricci curvature if and only if its Ricci curvature tensor S is not identically zero and*

$$(\nabla_U S)(X, Y) = \varphi(U)S(X, Y) - \varphi(JU)S(X, JY) \tag{29}$$

for a certain real 1-form φ *such that* $\hat{\varphi} = \varphi - \sqrt{-1}J\varphi$ *is a holomorphic 1-form.*

Proof. For any Kähler–Norden manifold, we have

$$(\hat{\nabla}_{\hat{U}}\hat{S})(\hat{X}, \hat{Y}) = \tfrac{1}{2}((\nabla_U S)(X, Y) - \sqrt{-1}(\nabla_U S)(X, JY)). \tag{30}$$

To find (30), with the help of (15) and (19), we compute

$$\begin{aligned}(\hat{\nabla}_{\hat{U}}\hat{Q})\hat{X} &= \hat{\nabla}_{\hat{U}}\hat{Q}\hat{X} - \hat{Q}\hat{\nabla}\hat{U}\hat{X} = \hat{\nabla}_{\hat{U}}\widehat{QX} - \hat{Q}\widehat{\nabla_U X}\\ &= (\nabla_U Q\hat{X}) - (Q\nabla_U\hat{X}) = ((\nabla_U Q)X)\hat{}\,,\end{aligned}$$

which gives $(\hat{\nabla}_{\hat{U}}\hat{Q})\hat{X} = ((\nabla_U Q)\hat{X})$. Next we show that the last formula and (5) imply

$$\begin{aligned}(\hat{\nabla}_{\hat{U}}\hat{S})(\hat{X}, \hat{Y}) &= \hat{g}((\hat{\nabla}_{\hat{U}}\hat{Q})\hat{X}, \hat{Y}) = \hat{g}(((\nabla_U Q)X)\hat{Y}, \hat{Y})\\ &= \tfrac{1}{2}(g((\nabla_U Q)X, Y) - \sqrt{-1}\, g((\nabla_U Q)X, JY)),\end{aligned}$$

which leads to (30).

Returning to the Ricci H-recurrence, we note that the defining condition (28) can be equivalently written as

$$(\hat{\nabla}_{\hat{U}}\hat{S})(\hat{X},\hat{Y}) = \hat{\varphi}(\hat{U})\hat{S}(\hat{X},\hat{Y}). \tag{31}$$

Now, writing $\hat{\varphi} = \varphi - \sqrt{-1}J\varphi$ and using (25) and (19), we find

$$\hat{\varphi}(\hat{U})\hat{S}(\hat{X},\hat{Y}) = \tfrac{1}{2}(\varphi(U)S(X,Y) - \varphi(JU)S(X,JY) - \sqrt{-1}\,(\varphi(U)S(X,JY) + \varphi(JU)S(X,Y))).$$

Finally, using the above and (30), we assert that (31) holds if and only if (29) is fulfilled. □

Examples of Kähler–Norden manifolds of H-recurrent Ricci curvature will be given in Section 8.

6. Holomorphically projective curvature

The holomorphically projective (in short, H-projective) curvature (real) tensor P of a Kähler–Norden manifold of real dimension $n = 2m > 2$ is defined by [23,10,18]

$$P(X,Y) = R(X,Y) - \frac{1}{n-2}(X \wedge_S Y - JX \wedge_S JY). \tag{32}$$

The tensor P is an invariant of the holomorphically projective transformations. On the other hand, by analogy to the classical theory of the (real) projective curvature (see e.g. [3]), consider the standard projective curvature tensor $\hat{P}$ of the connection $\hat{\nabla}$ given by

$$\hat{P}(\hat{X},\hat{Y}) = \hat{R}(\hat{X},\hat{Y}) - \frac{1}{m-1}\hat{X} \wedge_{\hat{S}} \hat{Y}. \tag{33}$$

By definition, $\hat{P}$ is a holomorphic tensor filed. Making straightforward computations, in which formulas (33), (17), (21) with $A = S$, and (32) should be used, one can check that the tensors $\hat{P}$ and P are strictly related. Namely:

$$\hat{P}(\hat{X},\hat{Y})\hat{Z} = (P(X,Y)Z)\hat{} \tag{34}$$

holds.

A Kähler–Norden manifold is called H-projectively flat if its Levi–Civita connection can be locally holomorphically projectively transformed to a flat connection.

Theorem 5. *Let (M, J, g) be a Kähler–Norden manifold.*

(i) *When $\dim_{\mathbb{R}} M \geq 6$, the manifold is H-projectively flat if and only if $P = 0$, or equivalently $\hat{P} = 0$.*
(ii) *When $\dim_{\mathbb{R}} M = 4$, the tensor P (or equivalently $\hat{P}$) vanishes identically. In this case, the manifold is H-projectively flat if and only if its scalar curvature r is constant, or equivalently $\hat{r}$ is constant.*

Proof.

(i) This assertion is in fact a consequence of more general results concerning complex manifolds endowed with affine connections (cf. [20,23,10]) and formulas (33), (34).
(ii) Since the Riemann curvature tensor of a Kähler–Norden manifold is totally pure (see (16)), from Corollary 4.2 of Ivanov [10] it follows that P vanishes identically, and the manifold is H-projectively flat if and only if the Ricci tensor S is Codazzi, that is:

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0. \tag{35}$$

In view of (30), the condition (35) is equivalent to

$$(\hat{\nabla}_{\hat{X}} \hat{S})(\hat{Y}, \hat{Z}) - (\hat{\nabla}_{\hat{Y}} \hat{S})(\hat{X}, \hat{Z}) = 0. \tag{36}$$

On the other hand, since $\dim_{\mathbb{C}} M = 2$, for the Ricci tensor $\hat{S}$, we have $\hat{S} = (\hat{r}/2)\hat{g}$. Therefore, (36) holds if and only if

$$\mathrm{d}\hat{r}(\hat{X})\hat{g}(\hat{Y}, \hat{Z}) - \mathrm{d}\hat{r}(\hat{Y})\hat{g}(\hat{X}, \hat{Z}) = 0,$$

equivalently $\mathrm{d}\hat{r} = 0$, that is, $\hat{r}$ is a complex constant. Finally, since $\hat{r}$ is a holomorphic function and $\mathrm{Re}(\hat{r}) = r/2$, $\hat{r}$ is constant if and only if r is constant. □

A Kähler–Norden manifold is called to be of recurrent H-projective curvature if its tensor P is not identically zero and for a certain 1-form ψ:

$$\nabla P = \psi \otimes P.$$

It was shown by the author in [18] that this notion is not essential in the sense that any Kähler–Norden manifold of recurrent H-projective curvature is necessarily locally symmetric. So, we examine the following.

A Kähler–Norden manifold will be called to be of H-recurrent H-projective curvature if its tensor $\hat{P}$ is not identically zero and

$$\hat{\nabla} \hat{P} = \hat{\varphi} \otimes \hat{P} \tag{37}$$

for a certain holomorphic 1-form $\hat{\varphi}$

The next theorem shows that the class of Kähler–Norden manifolds of H-recurrent H-projective curvature reduces to the class of H-recurrent curvature. Precisely, we have the following theorem.

Theorem 6. *Let* (M, J, g) *be a Kähler–Norden manifold with* $\dim_{\mathbb{R}} M \geq 6$.

(i) *If* (M, J, g) *is of H-recurrent curvature, then it is H-projectively flat or of H-recurrent H-projective curvature*.
(ii) *If* (M, J, g) *is of H-recurrent H-projective curvature, then it is of H-recurrent curvature*.

Proof.

(i) Let us assume that the manifold is of H-recurrent curvature, so that (22) holds and $\hat{R} \neq 0$. We note easily that (22) always implies (28). Next differentiating (33) covariantly and applying both of the relations, we find (37). Now, we have two possibilities:
(i_1) $\hat{P}$ is identically zero, i.e. the H-projective flatness (however, in this case the manifold is locally symmetric, which is not excluded in general; cf. [18]);
(i_2) $\hat{P} \neq 0$ at every point and then the manifold is of H-recurrent H-projective curvature.
(ii) This part of our theorem can be proved by formal repeating certain ideas from the papers [14,8], which concern Riemannian manifolds with recurrent projective curvature. Exactly, assuming the condition (37) and $\hat{P} \neq 0$, similarly like in Theorem 3.6 of [14], it can be shown that the recurrence form $\hat{\varphi}$ is closed. Next, like in [8], one shows that (22) must hold. $\hat{R} \neq 0$ since $\hat{P} \neq 0$. □

7. Four-dimensional Kähler–Norden manifolds

Theorem 7. *A Kähler–Norden manifold of real dimension 4 is of H-recurrent curvature on the set, where the holomorphic scalar curvature $\hat{r}$ is not zero with the recurrence form $\hat{\varphi}$ given by $\hat{\varphi} = \mathrm{d}\hat{r}/\hat{r}$, or equivalently $\hat{\varphi} = \varphi - \sqrt{-1}J\varphi$ with $\varphi = (1/2)\mathrm{d}\ \ln(r^2 + r^{*2})$.*

Proof. Since $\dim_{\mathbb{C}} M = 2$, we may express $\hat{R}$ in the form

$$\hat{R}(\hat{X}, \hat{Y}) = \tfrac{1}{2}\hat{r}\hat{X} \wedge_{\hat{g}} \hat{Y}.$$

This then leads to the following expression

$$(\hat{\nabla}_{\hat{U}} \hat{R})(\hat{X}, \hat{Y}) = \tfrac{1}{2}\mathrm{d}\hat{r}(\hat{U})\hat{X} \wedge_{\hat{g}} \hat{Y}.$$

The last two equalities give (22) with $\hat{\varphi} = \mathrm{d}\hat{r}/\hat{r}$ at points where $\hat{r} \neq 0$. Using formula (20), we find

$$\mathrm{Re}(\hat{\varphi}) = \tfrac{1}{2}\mathrm{d}\ \ln(r^2 + r^{*2}).$$

Therefore, we can write $\hat{\varphi} = \varphi - \sqrt{-1}J\varphi$, where $\varphi = (1/2)\mathrm{d}\ \ln(r^2 + r^{*2})$. □

Theorem 8. *A Kähler–Norden manifold of real dimension 4 is (locally) conformally flat if and only if its holomorphic scalar curvature is constant pure imaginary* ($r = 0$ *and* $r^* =$ constant).

Proof. Since $\dim_{\mathbb{C}} M = 2$, like in the previous proof, we have

$$\hat{R}(\hat{X}, \hat{Y})\hat{Z} = \tfrac{1}{2}\,\hat{r}\,(\hat{X} \wedge_{\hat{g}} \hat{Y})\hat{Z}. \tag{38}$$

To transform the right hand side of (38), we will use (9) and the formula

$$J(X \wedge_g Y - JX \wedge_g JY) = JX \wedge_g Y + X \wedge_g JY, \tag{39}$$

which can be obtained by a straightforward computation. Namely, using (21) with $A = g$, we find

$$\tfrac{1}{2}\,\hat{r}\,(\hat{X} \wedge_{\hat{g}} \hat{Y})\hat{Z} = \tfrac{1}{4}\,\hat{r}\,((X \wedge_g Y)Z - (JX \wedge_g JY)Z)\hat{,}$$

which in virtue of (9) with $f = \hat{r}$, $W = (X \wedge_g Y)Z - (JX \wedge_g JY)Z$, and the formulas (20), (39), can be transformed into

$$\tfrac{1}{2}\,\hat{r}\,(\hat{X} \wedge_{\hat{g}} \hat{Y})\hat{Z} = \tfrac{1}{8}((r(X \wedge_g Y - JX \wedge_g JY) - r^*(JX \wedge_g Y + X \wedge_g JY))Z)\hat{.} \tag{40}$$

Now, regarding (38), (17) and (40), we obtain for the curvature tensor

$$R(X, Y) = \tfrac{1}{8} r(X \wedge_g Y - JX \wedge_g JY) - \tfrac{1}{8} r^*(JX \wedge_g Y + X \wedge_g JY). \tag{41}$$

Consequently, for the Ricci operator:

$$Q = (\tfrac{1}{4} r)I - (\tfrac{1}{4} r^*)J. \tag{42}$$

Recall that the manifold is conformally flat if and only if $C = 0$, where C is the Weyl conformal curvature tensor:

$$C(X, Y) = R(X, Y) - \tfrac{1}{2}(QX \wedge_g Y + X \wedge_g QY - (\tfrac{1}{3} r)X \wedge_g Y).$$

In virtue of (41) and (42), the conformal curvature tensor takes the shape

$$C(X, Y) = \tfrac{1}{24} r(X \wedge_g Y - 3JX \wedge_g JY).$$

Therefore, $C = 0$ if and only if $r = 0$, as required. □

8. Examples

Let $m \in \mathbb{N}$, $m > 3$, and the Greek indices run through the set $\{2, 3, \ldots, m-1\}$. Let $\hat{g}$ be the holomorphic Riemannian metric which is defined on an open subset U of the complex space $\mathbb{C}^m$ by

$$\hat{g} = f\,\mathrm{d}z^1 \otimes \mathrm{d}z^1 + \sum k_{\alpha\beta}\,\mathrm{d}z^\alpha \otimes \mathrm{d}z^\beta + \mathrm{d}z^1 \otimes \mathrm{d}z^m + \mathrm{d}z^m \otimes \mathrm{d}z^1, \tag{43}$$

where the sum concerns the repeated indices, f is a holomorphic function on U and $k_{\alpha\beta}$ are complex constants such that the $(m-2)$-by-$(m-2)$ matrix $[k_{\alpha\beta}]$ is symmetric and non-degenerate. The metric (43) is the holomorphic version of the so-called Walker's type metric occurred in [21], Section 9; see also the monograph [17].

In the sequel, we specify the function f to obtain various classes of holomorphic Riemannian metrics.

(i) Let us suppose

$$f(z^1,\ldots,z^m)=p(z^1)\sum a_{\alpha\beta}z^\alpha z^\beta+\sum b_\alpha z^\alpha, \tag{44}$$

where $a_{\alpha\beta}$ are complex constants such that the $(m-2)$-by-$(m-2)$ matrix $[a_{\alpha\beta}]$ is non-zero and symmetric, b_α are arbitrary complex constants and p is a function depending on one complex variable only, which is non-zero and holomorphic on an open subset U_1 of the complex line $\mathbb{C}$. Then on the open subset $U=U_1\times\mathbb{C}^{m-1}\subset\mathbb{C}^m$, the metric (43) with f given by (44) is of H-recurrent curvature with $\hat{\varphi}=(p'/p)\mathrm{d}z^1$ as its recurrence form (motivated by Walker [21], Section 9).

(ii) Suppose that

$$f(z^1,\ldots,z^m)=2z^2\sum a_\alpha z^\alpha, \tag{45}$$

where a_α are complex constants such that

$$\sum |a_\alpha|^2>0 \quad \text{and} \quad \sum k^{\alpha\beta}a_\alpha a_\beta=0,$$

$[k^{\alpha\beta}]$ being the inverse matrix $[k_{\alpha\beta}]^{-1}$. Then the metric (43) with f given by (45) is of H-recurrent Ricci curvature with $\hat{\varphi}=\left(\sum_\alpha a_\alpha z^\alpha\right)\,\mathrm{d}z^1$ as its recurrence form (motivated by Olszak [15], Example 1).

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